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Random walks and Schrödinger's equation in $(2 + 1)$ dimensions

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Abstract. Some recent results have shown that in addition to its role in quantum mechanics, the Schrödinger free particle equation in $(1 + 1)$ dimensions describes second-order effects in ensembles of lattice random walks. This alternative classical context for Schrödinger's equation is independent of its role in quantum mechanics. In this paper we extend this result to include the case of Schrödinger's equation in $(2 + 1)$ dimensions for a particle in a smooth bounded potential. The extension suggests that the new classical context of Schrödinger's equation is quite general.

0. Introduction

It would be difficult to overstate the importance of Schrodinger's equation in our current understanding of physics and chemistry. However, inspite of a long history of great success in the precise prediction of microscopic phenomena, the equation itself remains an enigma. This is largely due to the historical association of the equation with quantum mechanics. Within quantum mechanics, 'wavefunction' solutions of Schrödinger's equation are mathematical objects which have no known physical counterpart. Wavefunctions facilitate the calculation of 'observables' but are not themselves observable.

The absence of an acceptable microscopic model for quantum mechanics has given rise to a division within physics centred around the original Bohr–Einstein debate. In that debate Einstein maintained that without a microscopic model, quantum mechanics was incomplete. Bohr, on the other hand, argued that quantum mechanics was complete as a theory and that pursuit of a more fundamental description was futile. The debate, which remains unresolved, moved from the arena of thought experiments into the realm of real experiments due partly to the work of J S Bell [1]. A very interesting article containing Bell's final views on the subject can be found in [2].

There has been a recent revival of interest in the interpretation of quantum mechanics within the physics community. The volumes by El Naschie, Rossler and Prigogine [3] provide some examples of very recent work in the field. The books by Nelson [4] and Nagasawa [5] describe Stochastic approaches to quantum mechanics and Nottale [6] considers a fractal spacetime approach. A summary of Euclidean quantum mechanics may be found in the recent article by Zambrini [7] and the book by Holland [8] describes the current state of the de Broglie–Bohm formulation. Direct antecedents of this work can be found in [9–17].

For all practical purposes the uncertainty about interpretations of quantum mechanics can be (and generally is) completely ignored by those who use quantum mechanics to

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describe nature. In practice one just solves Schrödinger's equation and uses the solution to calculate relevant observables. There is then a temptation to conclude that the Schrödinger equation *is* quantum mechanics, and that as a result, Schrödinger's equation itself has no microscopic model. Part of the purpose of this paper is to illustrate the fact that this latter conclusion is incorrect. That is, we shall show that Schrödinger's equation, which in the quantum context represents a particle in a smooth bounded potential in $(2+1)$ dimensions, also has a microscopic model completely within the classical mechanics of random walks. In the new context the solutions of Schrödinger's equation represent the continuum limit of a property of ensembles of Brownian particles.

The difficulty with deriving Schrödinger's equation from classical physics is that classical probability densities obey dissipative dynamics, whereas the 'probability amplitude' of Schrödinger's equation does not. Crossing the bridge between these two qualitatively different behaviours is mathematically easy and may be accomplished by a formal analytic continuation (e.g. $t \to it$ takes the diffusion equation to the free particle Schrödinger equation or the Wiener Integral to the Path integral [18]). Physically, however, the analytic continuation is difficult to interpret (e.g. real time versus imaginary time) and microscopic models which underlie classical probabilistic systems are qualitatively changed by the analytic continuation (e.g. Brownian motion versus reversible diffusions).

In this article we take a rather different track than is usual. The underlying microscopic model we use is a simple random walk model whose probabilistic description is completely classical. The dynamics are dissipative and particle densities obey the diffusion equation. We do not change these dynamics in order to 'see' Schrödinger's equation, we just examine the dynamics more closely than is usual. Schrödinger's equation then appears as a description of second-order effects in ensembles of these diffusing particles. The reversible dynamics associated with Schrödinger's equation reflect an intrinsic symmetry, inherent in lattice random walks, which is not seen at the level of particle density in the continuum limit. In this context the real and imaginary parts of the solutions of Schrödinger's equation are observable properties of *ensembles* of random walks in the same way that solutions of the diffusion equation are real observable properties of such ensembles. In this sense we have a classical microscopic model of Schrödinger's equation which is as direct as the random walk model of diffusion. There is then no difficulty in interpreting solutions of Schrödinger's equation in this context. However, the price paid for the objective reality of the underlying microscopic model is that it cannot correspond directly to individual particles of nature. None of the individual particles in our formulation have any of the aspects of waveparticle duality which would be required to imitate the particles of nature. Interference effects are seen only at the level of ensembles of particles, not at a single particle level.

The model we consider is a generalization of models which have been studied in $(1+1)$ dimensions [14, 17]. In these models, we can obtain both Schrödinger's equation and the diffusion equation directly within classical statistical mechanics, by projection. Figure 1 represents the situation schematically. Random walks on a lattice provide a microscopic model for the diffusion equation and the usual route to Schrödinger's equation then involves a FAC (e.g. $t \to it$). This provides an equation for wavefunctions without identifying what objective properties of the random walk model, if any, they really describe. The new route from random walks to the Schrödinger equation involves only a projection so that the solutions in this case have direct counterparts in ensembles of random walks.

In this work we extend the picture of figure 1 to include $(2+1)$ dimensions and a smooth bounded external field. In section 1 we consider the $(1+1)$-dimensional case in which walks take place in an external field, following [16]. Section 2 extends this to $(2+1)$ dimensions.

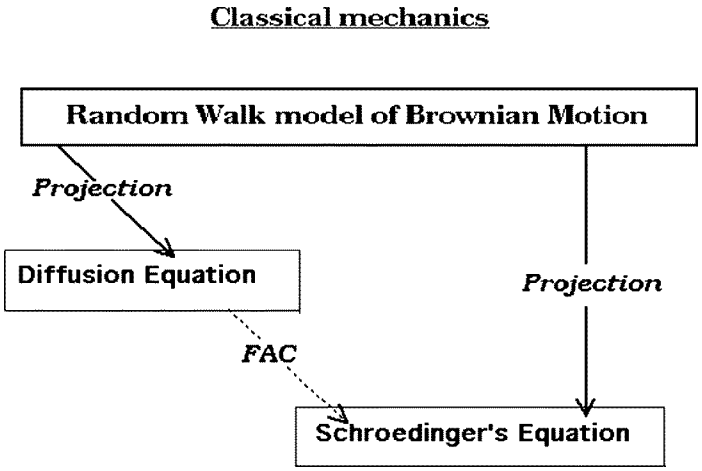

Figure 1. The relation between the lattice random walk model of Brownian motion, the diffusion equation, and Schrödinger's equation. Until recently the only route to Schrödinger's equation involved a FAC. This FAC was a formal step which removed the random walk model as a truly microscopic model of the Schrödinger equation. The new route to the Schrödinger equation is a projection in which there is no formal step. Taking into account this new route, both equations have the same microscopic model and are simply different projections of the same system.

Table 1.

State	Direction	Spin
1	Right	1
2	Left	1
3	Right	−1
4	Left	−1

1. Walks in $(1+1)$ dimensions

We consider discrete random walks on a space time lattice with lattice spacing δ and ϵ in x and t, respectively. We shall keep track of a particle's history on the lattice by recording its state at each time step. We shall distinguish four possible states. States one and three correspond to right-moving particles and states two and four to left-moving particles. A particle starting in state one changes to state two at the first direction reversal, state three at the second reversal, state four at the third reversal and back to one at the fourth reversal.

As an alternative labelling of states we associate an Ising spin variable $\sigma = \pm 1$ with each state. We say that states one and two have spin $\sigma = +1$ and states three and four have spin $\sigma = -1$. This gives us a convenient means of distinguishing different states which correspond to the same direction (see table 1).

Note here that the 'state of a particle' really refers to a state of its trajectory. We are not adding any new property to the point particles by considering these four different states, we are only recording more information about the trajectory than is usual.

Let $p_\mu(m\delta, s\epsilon)\delta$ $(\mu = 1, 2, 3, 4)$ be the probability that a particle is in state μ at the spacetime point $x = m\delta$, $t = s\epsilon$ $(m = 0, \pm 1, \pm 2, \ldots; s = 0, 1, \ldots)$. The difference

equations for the p_μ are then

$$\begin{aligned} p_1(m\delta,(s+1)\epsilon) &= \alpha p_1((m-1)\delta, s\epsilon) + \beta p_4((m+1)\delta, s\epsilon) \\ p_2(m\delta,(s+1)\epsilon) &= \alpha p_2((m+1)\delta, s\epsilon) + \beta p_1((m-1)\delta, s\epsilon) \\ p_3(m\delta,(s+1)\epsilon) &= \alpha p_3((m-1)\delta, s\epsilon) + \beta p_2((m+1)\delta, s\epsilon) \\ p_4(m\delta,(s+1)\epsilon) &= \alpha p_4((m+1)\delta, s\epsilon) + \beta p_3((m-1)\delta, s\epsilon) \end{aligned} \tag{1}$$

where $\alpha + \beta = 1$. Here, α is the probability that a particle maintains its direction at the next time step; whereas, β is the probability that a particle will change its direction at the next time step. Equation (1) is the master equation for the ensemble of random walks. We shall shortly let α vary according to an external field. We impose the condition

$$\sum_{\mu=1}^{4} \sum_{m=-\infty}^{+\infty} p_\mu(m\delta, s\epsilon)\delta = 1 \tag{2}$$

which establishes the fact that the probability that a particle is somewhere on the lattice at a given time is 1.

The governing equations (1) have a straightforward interpretation. The first equation in (1) implies that the probability $p_1\delta$ that the particle leaves the node $(m\delta,(s+1)\epsilon)$ in state 1 is equal to the sum of two proabilities: $\alpha p_1\delta$—the probability that the particle leaves the node $((m-1)\delta, s\epsilon)$ in state 1 and remains in this state when it leaves $(m\delta,(s+1)\epsilon)$, $\beta p_4\delta$—the probability that the particle leaves $((m+1)\delta, s\epsilon)$ in state 4 and changes to state 1 when it leaves the node $(m\delta,(s+1)\epsilon)$. Once the initial conditions are given, (1) has a unique solution.

The parameters ϵ and δ are related by the requirement that in the diffusive continuum limit $\delta/(2\epsilon) \to D$ as $\delta \to 0$ where D is the diffusion constant. Hence, we have, for small δ,

$$\frac{\delta^2}{2\epsilon} = D + \mathrm{O}(\delta) \qquad \text{or} \qquad \epsilon = \frac{\delta^2}{2D} + \mathrm{O}(\delta^3). \tag{3}$$

To express (1) in matrix form, consider the shift operators $E_x^{\pm 1}$ and E_t such that

$$E_x^{\pm 1} p_i(m\delta, s\epsilon) = p_i((m \pm 1)\delta, s\epsilon) \qquad E_t\, p_i(m\delta, s\epsilon) = p_i(m\delta,(s+1)\epsilon). \tag{4}$$

Then equation (1) becomes

$$E_t \boldsymbol{p}(m\delta, s\epsilon) = \mathcal{E}_x \boldsymbol{p}(m\delta, s\epsilon) \tag{5}$$

where $\boldsymbol{p}(m\delta, s\epsilon) = [p_1(m\delta, s\epsilon), \ldots, p_4(m\delta, s\epsilon)]^T$ and

$$\mathcal{E}_x = \begin{bmatrix} \alpha E_x^{-1} & 0 & 0 & \beta E_x \\ \beta E_x^{-1} & \alpha E_x & 0 & 0 \\ 0 & \beta E_x & \alpha E_x^{-1} & 0 \\ 0 & 0 & \beta E_x^{-1} & \alpha E_x \end{bmatrix}. \tag{6}$$

Now consider the change of variables

$$\begin{aligned} \tilde{z}_1 &= p_1 + p_2 + p_3 + p_4 \qquad & \tilde{z}_3 &= p_1 - p_3 \\ \tilde{z}_2 &= (p_1 + p_3) - (p_2 + p_4) \qquad & \tilde{z}_4 &= p_2 - p_4. \end{aligned} \tag{7}$$

In matrix notation, we have

$$\tilde{\boldsymbol{z}} = [\tilde{z}_1, \ldots, \tilde{z}_4]^T = R\boldsymbol{p} \tag{8}$$

with

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}. \tag{9}$$

From table 1, $\tilde{z}_1(m\delta, s\epsilon)\delta$ is the probability that a particle leaves $(m\delta, s\epsilon)$ in either direction and in any spin state, $\tilde{z}_2\delta$ is the difference in the probabilites that a particle leaves $(m\delta, s\epsilon)$ to the right and the left, and $\tilde{z}_3\delta = (+1)p_1\delta + (-1)p_3\delta$ is the expected spin of a particle leaving $(m\delta, s\epsilon)$ to the right. Similarly, $\tilde{z}_4\delta$ is the expected spin of a particle leaving $(m\delta, s\epsilon)$ to the left.

It is worth noting at this point that the change of variables (7) does not change the object we are describing (i.e. ensembles of random walkers obeying the master equation (1)). The new variables are not probabilities, but they are expectations of simple 'counting' variables over well-defined ensembles of walks. As such they represent observable features of the ensembles of walks.

In the new variables equation (5) becomes

$$E_t\tilde{\boldsymbol{z}}(m\delta, s\epsilon) = R\mathcal{E}_x R^{-1}\tilde{\boldsymbol{z}}(m\delta, s\epsilon). \tag{10}$$

As in [17], the variables may be scaled with a change of variables from $\tilde{\boldsymbol{z}}$ to $\boldsymbol{z} = (z_1, z_2, z_3, z_4)^T$ where we choose a normalization appropriate to the continuum limit; that is,

$$z_i = \tilde{z}_i \qquad z_{i+2} = 2^{s/2}\tilde{z}_{i+2} \qquad (i = 1, 2). \tag{11}$$

Then (10) becomes

$$E_t\boldsymbol{z}(m\delta, s\epsilon) = \mathcal{E}'_x\boldsymbol{z}(m\delta, s\epsilon) \tag{12}$$

where

$$\mathcal{E}'_x = \begin{bmatrix} I_2 & 0 \\ 0 & \sqrt{2}I_2 \end{bmatrix} R\mathcal{E}_x R^{-1} = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \tag{13}$$

$$\begin{aligned} B_{11} &= \frac{1}{2}\begin{bmatrix} (E_x + E_x^{-1}) & (E_x^{-1} - E_x) \\ (\beta - \alpha)(E_x^{-1} - E_x) & (\alpha - \beta)(E_x^{-1} + E_x) \end{bmatrix} \\ B_{22} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 2\alpha E_x^{-1} & -2\beta E_x \\ 2\beta E_x^{-1} & 2\alpha E_x \end{bmatrix} \end{aligned} \tag{14}$$

and I_2 is the (2×2) identity matrix.

As in the free-particle case discussed in [17], the shift matrix $\mathcal{E}'_x$ is block diagonal so that we may analyse (z_1, z_2) and (z_3, z_4) separately. However, before doing so we put in a potential field through α. Here we imagine the lattice walkers choose their next state according to a canonical ensemble in which a smooth bounded potential $v(x)\epsilon$ acts like an energy. That is, suppose

$$\alpha = \frac{\mathrm{e}^{-v(x)\epsilon}}{\mathrm{e}^{-v(x)\epsilon} + \mathrm{e}^{v(x)\epsilon}} \tag{15}$$

so that

$$\alpha = \tfrac{1}{2}(1 - v(x)\epsilon) + \mathrm{O}(\epsilon^2) \qquad \beta = \tfrac{1}{2}(1 + v(x)\epsilon) + \mathrm{O}(\epsilon^2). \tag{16}$$

For small δ, both α and β are equal to $\frac{1}{2} + \mathrm{O}(\delta^2)$.

We wish to approximate the solution of (12) for small δ by solutions of partial differential equations and we start with the first block of equations in (12). Suppose we are interested

in (x, t) in a neighbourhood of a fixed point (X, T) in spacetime. Given δ and ϵ, we select the node $(m\delta, s\epsilon)$ as $(M\delta, S\epsilon)$ such that

$$M\delta \leqslant X \leqslant (M+1)\delta \qquad S\epsilon \leqslant T < (S+1)\epsilon. \tag{17}$$

We start with $E_x^{\pm 1} z_i(M\delta, S\epsilon) = z_i(M\delta \pm \delta, S\epsilon)$ and expand $z_i(M\delta \pm \delta, S\epsilon)$ in a power series in δ to obtain

$$E_x^{\pm 1} = 1 \pm \delta \frac{\partial}{\partial x} + \frac{1}{2}\delta^2 \frac{\partial^2}{\partial x^2} + \mathrm{O}(\delta^3). \tag{18}$$

Similarly, from the expansion of $E_t z_i(M\delta, S\epsilon) = z_i(M\delta, S\epsilon + \epsilon)$, we have

$$E_t = 1 + \epsilon \frac{\partial}{\partial t} + \mathrm{O}(\epsilon^2). \tag{19}$$

We apply these expansions to the first block of equations in (12); namely, $E_t[z_1, z_2]^T = B_{11}[z_1, z_2]^T$. Substituting the expansions into this equation and using (16), we have

$$\frac{\partial}{\partial t} z_1(M\delta, S\epsilon) = D \frac{\partial^2}{\partial x^2} z_1(M\delta, S\epsilon) + \mathrm{O}(\delta) \qquad z_2 = \mathrm{O}(\delta). \tag{20}$$

Thus, $z_i(M\delta, S\epsilon) = z_i^*(M\delta, S\epsilon) + \mathrm{O}(\delta)$ where $z_1^*(x, t)$ is a solution of the diffusion equation $(z_1^*)_t = D(z_1^*)_{xx}$ and $z_2^*(x, t) = 0$. This is expected on physical grounds because the potential only affects the local mean free path and does not favour either direction. Since the mean free path is zero in the continuum limit the z_1 does not contain the (finite) potential in this limit.

The second block of equations in (12) to consider is $E_t \Phi(m\delta, s\epsilon) = B_{22}\Phi(m\delta, s\epsilon)$ where $\Phi = [z_3, z_4]^T$. As discussed in detail in [17], we cannot work directly with this equation if we wish to approximate Φ for small δ by a continuous function. To see this, we have, from (14) and (18), $B_{22} = V + \mathrm{O}(\delta)$ where V is defined in (24). Thus, $E_t \Phi(m\delta, s\epsilon) = \Phi(m\delta, (s+1)\epsilon) = V\Phi(m\delta, s\epsilon) + \mathrm{O}(\delta)$ and, in k steps on the lattice, $\Phi(m\delta, (s+k)\epsilon) = V^k \Phi(m\delta, s\epsilon) + \mathrm{O}(\delta)$. Since V is a rotation matrix with angle of rotation $\pi/4$, $V^8 = I_2$. Hence, $\Phi(m\delta, (s+8l)\epsilon) - \Phi(m\delta, s\epsilon) = \mathrm{O}(\delta)$ $(l = 0, 1, \ldots)$ for any s so that Φ can be approximated by a continuous function for small δ provided we consider Φ defined on the time steps $s + 8l$. In the sequel, we restrict Φ in most cases to the time steps $s = 8l$ and we indicate later what changes occur if we restrict Φ to $s = k + 8l$ $(k = 1, \ldots, 7)$. That is, given δ, ϵ and a fixed point (X, T), we select the node $(M\delta, S\epsilon)$ on the lattice where

$$M\delta \leqslant X \leqslant (M+1)\delta \qquad S\epsilon \leqslant T < (S+8)\epsilon \qquad S = 0 \bmod 8 \tag{21}$$

and approximate Φ by a solution of a partial differential equation. Applying E_t to $E_t\Phi = B_{22}\Phi$ seven times, we obtain

$$E_t^8[z_3(M\delta, S\epsilon), z_4(M\delta, S\epsilon)]^T = B_{22}^8[z_3(M\delta, S\epsilon), z_4(M\delta, S\epsilon)]^T \tag{22}$$

as the equations we use to approximate z_i. Rewriting B_{22} as an expansion in powers of δ we obtain

$$B_{22} = V + B\delta \frac{\partial}{\partial x} + \frac{1}{2}\delta^2 \left(V \frac{\partial^2}{\partial x^2} - \frac{v(x)}{D} V^T \right) + \mathrm{O}(\delta^3) \tag{23}$$

where B_{22} and its powers are expressed in terms of the matrices

$$\begin{aligned} V &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad B = -\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ C &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{24}$$

The following results can be readily established:

$$B_{22}^2 = C + \sqrt{2}B\delta\frac{\partial}{\partial x} + \left(\sqrt{2}V\frac{\partial^2}{\partial x^2} - I_2\frac{v(x)}{D}\right)\delta^2 + \mathrm{O}(\delta^3) \tag{25}$$

$$B_{22}^4 = -I_2 + 2C\left(\frac{\partial^2}{\partial x^2} - \frac{v(x)}{D}\right)\delta^2 + \mathrm{O}(\delta^3) \tag{26}$$

$$B_{22}^8 = I_2 - 4C\left(\frac{\partial^2}{\partial x^2} - \frac{v(x)}{D}\right)\delta^2 + \mathrm{O}(\delta^3). \tag{27}$$

Substituting the expansions (27) and $E_t^8 = 1+8\epsilon\frac{\partial}{\partial t}+\mathrm{O}(\epsilon^2)$ from (19) into equation (22), we have

$$\frac{\partial}{\partial t}\begin{bmatrix} z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 & D\frac{\partial^2}{\partial x^2} - v \\ -D\frac{\partial^2}{\partial x^2} + v & 0 \end{bmatrix}\begin{bmatrix} z_3 \\ z_4 \end{bmatrix} + \mathrm{O}(\delta) \tag{28}$$

which can be expressed in the complex form

$$\mathrm{i}\frac{\partial}{\partial t}(z_4 + \mathrm{i}z_3) = \left(-D\frac{\partial^2}{\partial x^2} + v\right)(z_4 + \mathrm{i}z_3) + \mathrm{O}(\delta) \tag{29}$$

where the functions are evaluated at the point $(M\delta, S\epsilon)$ on the lattice. Thus, $z_4(M\delta, S\epsilon) + \mathrm{i}z_3(M\delta, S\epsilon) = z_4^*(M\delta, S\epsilon)+\mathrm{i}z_3^*(M\delta, S\epsilon)+\mathrm{O}(\delta)$ where $\psi = z_4^*(x,t)+\mathrm{i}z_3^*(x,t)$ is a solution of the Schrödinger equation $\mathrm{i}\psi_t = -D\psi_{xx} + v(x)\psi$ for a particle in a scalar potential $v(x)$. However, in this context the equations apply to ensembles or Brownian particles on a discrete lattice with external field $v(x)$. Finally, we note that if we had used the equations on the time steps $s = k+8l$ $(k = 1, \ldots, 7)$ $(l = 0, 1, \ldots)$ then the equation we obtain is (28) multiplied by V^k, and hence, $V^k[z_3, z_4]^T$ is a solution of (28); consequently, $[\mathrm{i}, 1]V^k[z_3, z_4]^T$ is a solution of Schrödinger's equation. In conclusion, on the lattice we have

$$\boldsymbol{p} = \boldsymbol{r}_1 z_1^* + \frac{1}{2^{s/2}}(\boldsymbol{r}_3 z_3^* + \boldsymbol{r}_4 z_4^*) + \mathrm{O}(\delta) \qquad (s = 0 \bmod 8) \tag{30}$$

where $\boldsymbol{r}_i$ is the ith column vector of R^{-1}. z_1^* is a solution of the diffusion equation and $z_4^* + \mathrm{i}z_3^*$ is a solution of the Schrödinger equation.

Before leaving the $(1 + 1)$-dimensional case, a qualitative feature of the calculation which we emphasize is the following. Although we started out with classical dissipative dynamics described by (1) and (20), equation (28) describes a non-dissipative feature of the ensemble of walks. The representation (29) of equations (28) is not a formal analytic continuation of either (20) or (28), it is an expression of (28) in a form which is recognizable as Schrödinger's equation. The wave aspects of Schrödinger's equation are already intrinsic to (28) and consequently to the ensemble of particles being described.

2. Walks in $(2 + 1)$ dimensions

We now proceed with a generalization of the above model to $(2+1)$ dimensions as follows. Assume that particles hop along diagonals on a square lattice with spacing δ in both directions. If the space axes are labelled x and y then at each time step every particle moves a distance $\pm\delta$ along both axes. The projection of a trajectory onto either axis will then be a simple binary random walk of the type examined in the first section. We then count trajectories on projection in the same manner that we did for the $(1 + 1)$ case. That is, each projected walk will be labelled using four states: two spin and two direction states.

The total number of states in the system will then be 16, four for each direction. As in the previous case we let $p_{\mu\nu}(m\delta, n\delta, s\epsilon)\delta^2$ be the probability that a particle leaves $(m\delta, n\delta)$

at time $s\epsilon$ in state (μ, ν) $(\mu, \nu = 1, \ldots, 4)$. Here, μ denotes the state of the projection on the x-axis and ν denotes the state of the projection on the y-axis. The difference equations analogous to (1) are

$$\begin{aligned} p_{11}(m\delta, n\delta, (s+1)\epsilon) &= \alpha^2 p_{11}((m-1)\delta, (n-1)\delta, s\epsilon) \\ &\quad +\alpha\beta p_{41}((m+1)\delta, (n-1)\delta, s\epsilon) \\ &\quad +\beta\alpha p_{14}((m-1)\delta, (n+1)\delta, s\epsilon) + \beta^2 p_{44}((m+1)\delta, (n+1)\delta, s\epsilon) \\ &\vdots \\ p_{44}(m\delta, n\delta, (s+1)\epsilon) &= \alpha^2 p_{44}((m+1)\delta, (n+1)\delta, s\epsilon) \\ &\quad +\alpha\beta p_{34}((m-1)\delta, (n+1)\delta, s\epsilon) \\ &\quad +\beta\alpha p_{43}((m+1)\delta, (n-1)\delta, s\epsilon) + \beta^2 p_{33}((n-1)\delta, (m-1)\delta, s\epsilon). \end{aligned} \tag{31}$$

If we arrange these difference equations in rows of four so that within blocks only the x-label μ changes, the shift matrix then has a simple structure. As in (4), we define $E_x^{\pm 1} p(m\delta, n\delta, s\epsilon) = p(m\delta \pm \delta, n\delta, s\epsilon)$, $E_y^{\pm 1} p(m\delta, n\delta, s\epsilon) = p(m\delta, n\delta \pm \delta, s\epsilon)$, and $E_t p(m\delta, n\delta, s\epsilon) = p(m\delta, n\delta, s\epsilon + \epsilon)$. We have

$$E_t \boldsymbol{p}(m\delta, n\delta, s\epsilon) = \mathbb{E}\boldsymbol{p}(m\delta, n\delta, s\epsilon) \tag{32}$$

where $\boldsymbol{p} = [p_{11}, p_{21}, \ldots, p_{44}]^T$,

$$\mathbb{E} = \begin{bmatrix} \alpha E_y^{-1}\mathcal{E}_x & 0 & 0 & \beta E_y \mathcal{E}_x \\ \beta E_y^{-1}\mathcal{E}_x & \alpha E_y \mathcal{E}_x & 0 & 0 \\ 0 & \beta E_y \mathcal{E}_x & \alpha E_y^{-1}\mathcal{E}_x & 0 \\ 0 & 0 & \beta E_y^{-1}\mathcal{E}_x & \alpha E_y \mathcal{E}_x \end{bmatrix} \tag{33}$$

and $\mathcal{E}_x$ is the (4×4) matrix defined by (6). $\mathbb{E}$ can be expressed in a compact form using the outer (direct) product of two matrices. Let $A = (a_{ij})$, $B = (b_{ij})$, C and D be any $(n \times n)$ matrices. Then we denote the outer product of A and B as $A \otimes B$ and define it as the $(n^2 \times n^2)$ matrix where the (i, j) block-entry is the $(n \times n)$ matrix $a_{ij}B$. The property of outer products that we use later is $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. With this definition, $\mathbb{E} = \mathcal{E}_y \otimes \mathcal{E}_x$ where $\mathcal{E}_y$ is defined by replacing x by y in $\mathcal{E}_x$.

In the one-dimensional case we changed variables using the matrix R of equation (4). Here the equivalent change of variables is accomplished by the matrix $R \otimes R$. To simplify the set of equations, we change variables from $\boldsymbol{p}$ to $\boldsymbol{w}$ where $\boldsymbol{p} = (R^{-1} \otimes R^{-1})\boldsymbol{w}$ to obtain

$$E_t \boldsymbol{w} = (R \otimes R)(\mathcal{E}_y \otimes \mathcal{E}_x)(R^{-1} \otimes R^{-1})\boldsymbol{w}. \tag{34}$$

We have used the property that $(R \otimes R)(R^{-1} \otimes R^{-1}) = I$. Using the property of the outer product, this equation becomes

$$E_t \boldsymbol{w} = (R\mathcal{E}_y R^{-1}) \otimes (R\mathcal{E}_x R^{-1})\boldsymbol{w}. \tag{35}$$

With the aid of a computer algebra system, we perform the following change of variables in order to simplify (35). Let $P_{i,j}$ be the (16×16) permutation matrix which is obtained by interchanging the i and j rows of I_{16} (I_k is the $(k \times k)$ identity matrix). In addition, let

$$Q = \begin{bmatrix} I_{12} & 0 \\ 0 & S \end{bmatrix} \qquad S = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \tag{36}$$

and let $\boldsymbol{w} = P\tilde{\boldsymbol{z}}$ where $P = P_{11,13}P_{12,14}P_{3,5}P_{4,6}Q$. The equations now have the form

$$E_t \tilde{\boldsymbol{z}} = P^{-1}(R\mathcal{E}_y R^{-1}) \otimes (R\mathcal{E}_x R^{-1})P\tilde{\boldsymbol{z}} \tag{37}$$

in which the matrix has a block-diagonal structure where the blocks on the diagonal are (4×4) matrices. The fourth block is also block-diagonal where the sub-blocks are (2×2) matrices. Furthermore, we scale the variables in a way similar to the one-dimensional case by the change of variables:

$$\begin{aligned} z_i &= \tilde{z}_i \qquad (i = 1, \ldots, 4) \\ z_i &= 2^{s/2}\tilde{z}_i \qquad (i = 5, \ldots, 12) \\ z_i &= 2^{s}\tilde{z}_i \qquad (i = 13, \ldots, 16). \end{aligned} \tag{38}$$

The final form of our equations is

$$E_t \boldsymbol{z} = \mathbb{B}\boldsymbol{z} \qquad \mathbb{B} = P^{-1}(\mathcal{E}'_y \otimes \mathcal{E}'_x)P \tag{39}$$

where $\mathbb{B} = (B_{ij})$ and B_{ij} are (4×4) matrices with $B_{ij} = 0$, $i \neq j$. In addition, $\mathcal{E}'_x$ is defined in (13) and $\mathcal{E}'_y$ is obtained from $\mathcal{E}'_x$ by replacing x by y.

We start with the first (4×4) block of equations in (36), $E_t[z_1, \ldots, z_4]^T = B_{11}[z_1, \ldots, z_4]^T$, and determine the asymptotic expansion for small δ for (x, y, t) in a neighbourhood of a fixed point (X, Y, T) where the node $(M\delta, N\delta, S\epsilon)$ is defined, as in (17), by

$$M\delta \leqslant X < (M+1)\delta \qquad M\delta \leqslant Y < (M+1)\delta \qquad S\epsilon \leqslant T < (S+1)\epsilon. \tag{40}$$

We now proceed to expand B_{11} in powers of δ. To do this, we determine $E_x^{\pm 1}$ as in (18) and $E_y^{\pm 1}$ which follows from (18) with x replaced by y. We can show that for $B_{11} = (b_{ij})$

$$b_{11} = 1 + \tfrac{1}{2}\nabla^2\delta^2 + \mathrm{O}(\delta^3) \qquad b_{ij} = \mathrm{O}(\delta) \qquad ((i, j) \neq (1, 1)). \tag{41}$$

Substituting, we have $z_i = \mathrm{O}(\delta)$ $(i = 2, 3, 4)$ and

$$\frac{\partial}{\partial t} z_1 = D\nabla^2 z_1 + \mathrm{O}(\delta) \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{42}$$

Thus, $z_i(M\delta, N\delta, S\epsilon) = z_i^*(M\delta, N\delta, S\epsilon) + \mathrm{O}(\delta)$ where $z_1^*(x, y, t)$ is a solution of the diffusion equation $\partial z_1^*/\partial t = D\nabla^2 z_1^*$ and $z_i^*(x, y, t) = 0$ $(i = 2, 3, 4)$.

For the remaining cases, we consider, for the same reason as in (25),

$$\begin{aligned} &M\delta \leqslant X < (M+1)\delta \qquad M\delta \leqslant Y < (M+1)\delta \\ &S\epsilon \leqslant T < (S+8)\epsilon \qquad S = 0 \bmod 8. \end{aligned} \tag{43}$$

For $E_t[z_5, \ldots, z_8]^T = B_{22}[z_5, \ldots, z_8]^T$ we consider $E_t^8[z_5, \ldots, z_8]^T = B_{22}^8[z_5, \ldots, z_8]^T$ and expand as before. Let $B_{22}^8 = (M_{ij})$ where M_{ij} are (2×2) matrix, then we have

$$M_{11} = I_2 + \begin{bmatrix} \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x^2} - \frac{v}{D} \\ -\frac{\partial^2}{\partial x^2} + \frac{v}{D} & \frac{\partial^2}{\partial y^2} \end{bmatrix} 4\delta^2 + \mathrm{O}(\delta^3) \qquad M_{ij} = \mathrm{O}(\delta) \qquad ((i, j) \neq (1, 1)). \tag{44}$$

Substituting, we have $z_7 = \mathrm{O}(\delta)$, $z_8 = \mathrm{O}(\delta)$ and

$$\frac{\partial}{\partial t}\begin{bmatrix} z_5 \\ z_6 \end{bmatrix} = \begin{bmatrix} D\frac{\partial^2}{\partial y^2} & D\frac{\partial^2}{\partial x^2} - v \\ -D\frac{\partial^2}{\partial x^2} + v & D\frac{\partial^2}{\partial y^2} \end{bmatrix} \begin{bmatrix} z_5 \\ z_6 \end{bmatrix} + \mathrm{O}(\delta). \tag{45}$$

In complex form,

$$\mathrm{i}\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial y^2}\right)(z_6 + \mathrm{i}z_5) = \left(-D\frac{\partial^2}{\partial x^2} + v\right)(z_6 + \mathrm{i}z_5) + \mathrm{O}(\delta). \tag{46}$$

Thus, $z_i(M\delta, N\delta, S\epsilon) = z_i^*(M\delta, N\delta, S\epsilon) + \mathrm{O}(\delta)$ where $z_7^*(x, y, t) = z_8^*(x, y, t) = 0$, and $\psi = z_6^*(x, y, t) + \mathrm{i}z_5^*(x, y, t)$ is a solution of a 'mixed' equation $\mathrm{i}\psi_t = \mathrm{i}D\psi_{yy} - D\psi_{xx} + v\psi$: diffusive in y and Schrödinger-like in x.

For the next case, we have $E_t^8[z_9,\ldots,z_{12}]^T = B_{33}^8[z_9,\ldots,z_{12}]^T$ and expand as before. Let $B_{33}^8 = (c_{ij})$ and expand as a power series in δ to obtain

$$\begin{aligned} &c_{11} = 1 + 4\delta^2\frac{\partial^2}{\partial x^2} + \mathrm{O}(\delta^3) \qquad c_{13} = 4\delta^2\left(\frac{\partial^2}{\partial y^2} - \frac{v}{D}\right) + \mathrm{O}(\delta^3) \\ &c_{31} = -c_{13} \qquad c_{33} = c_{11} \end{aligned} \tag{47}$$

and the remaining entries $c_{ij} = \mathrm{O}(\delta)$. In conclusion, $z_{10} = \mathrm{O}(\delta)$, $z_{12} = \mathrm{O}(\delta)$ and

$$\frac{\partial}{\partial t}\begin{bmatrix} z_9 \\ z_{11}\end{bmatrix} = \begin{bmatrix} D\frac{\partial^2}{\partial x^2} & D\frac{\partial^2}{\partial y^2} - v \\ -D\frac{\partial^2}{\partial y^2} + v & D\frac{\partial^2}{\partial x^2}\end{bmatrix}\begin{bmatrix} z_9 \\ z_{11}\end{bmatrix} + \mathrm{O}(\delta) \tag{48}$$

which can be expressed in the complex form

$$\mathrm{i}\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right)(z_{11} + \mathrm{i}z_9) = \left(-D\frac{\partial^2}{\partial y^2} + v\right)(z_{11} + \mathrm{i}z_9) + \mathrm{O}(\delta). \tag{49}$$

Thus, $z_i(M\delta, N\delta, S\epsilon) = z_i^*(M\delta, N\delta, S\epsilon) + \mathrm{O}(\delta)$ where $z_{10}^*(x,y,t) = z_{12}^*(x,y,t) = 0$ and $\psi = z_{11}^*(x,y,t) + \mathrm{i}z_9^*(x,y,t)$ is a solution of the mixed equation $\mathrm{i}\psi_t = \mathrm{i}D\psi_{xx} - D\psi_{yy} + v\psi$: diffusive in x and Schrödinger-like in y.

In our last (4×4) block,we have $E_t[z_{13},\ldots,z_{16}]^T = B_{44}[z_{13},\ldots,z_{16}]^T$ and we consider $E_t^8[z_{13},\ldots,z_{16}]^T = B_{44}^8[z_{13},\ldots,z_{16}]^T$. Let $B_{44}^8 = (d_{ij})$ and the power series in δ is

$$\begin{aligned} d_{12} &= -d_{21} = 4\delta^2 D\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) + \mathrm{O}(\delta^3) \\ d_{34} &= -d_{43} = 4\delta^2\left(-\nabla^2 + \frac{v}{D}\right) + \mathrm{O}(\delta^3) \end{aligned} \tag{50}$$

where the remaining entries are $d_{ij} = \mathrm{O}(\delta)$. In conclusion,

$$\frac{\partial}{\partial t}\begin{bmatrix} z_{13} \\ z_{14}\end{bmatrix} = \begin{bmatrix} 0 & D(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}) \\ -D(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}) & 0\end{bmatrix}\begin{bmatrix} z_{13} \\ z_{14}\end{bmatrix} + \mathrm{O}(\delta) \tag{51}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} z_{15} \\ z_{16}\end{bmatrix} = \begin{bmatrix} 0 & -D\nabla^2 + 2v \\ D\nabla^2 - 2v & 0\end{bmatrix}\begin{bmatrix} z_{15} \\ z_{16}\end{bmatrix} + \mathrm{O}(\delta). \tag{52}$$

In complex form, these equations are

$$\mathrm{i}\frac{\partial}{\partial t}(z_{13} + \mathrm{i}z_{14}) = \left(-D\frac{\partial^2}{\partial y^2} + D\frac{\partial^2}{\partial x^2}\right)(z_{13} + \mathrm{i}z_{14}) + \mathrm{O}(\delta) \tag{53}$$

$$\mathrm{i}\frac{\partial}{\partial t}(z_{15} + \mathrm{i}z_{16}) = (-D\nabla^2 + 2v)(z_{15} + \mathrm{i}z_{16}) + \mathrm{O}(\delta). \tag{54}$$

Thus, $z_i(M\delta, N\delta, S\epsilon) = z_i^*(M\delta, N\delta, S\epsilon) + \mathrm{O}(\delta)$ where $\psi = z_{15}^*(x,y,t) + \mathrm{i}z_{16}^*(x,y,t)$ is a solution of Schrödinger's equation $\mathrm{i}\psi_t = (-D\nabla^2 + 2v)\psi$; whereas, $\psi = z_{13}^*(x,y,t) + \mathrm{i}z_{14}^*(x,y,t)$ is a solution of the mixed equation $\mathrm{i}\psi_t = -D\psi_{yy} + D\psi_{xx}$: Schrödinger-like in y and conjugate Schrödinger-like in x.

In conclusion, we have on the lattice

$$\boldsymbol{p} = \sum_{i=1}^{4}\boldsymbol{r}_i z_i^* + \frac{1}{2^{s/2}}\sum_{i=5}^{12}\boldsymbol{r}_i z_i^* + \frac{1}{2^s}\sum_{i=13}^{16}\boldsymbol{r}_i z_i^* + \mathrm{O}(\delta) \qquad (s = 0 \bmod 8) \tag{55}$$

where $\boldsymbol{r}_i$ is the ith column vector of $(R^{-1}\otimes R^{-1})P$. z_1^* is a solution of the diffusion equation; $z_{13}^* + \mathrm{i}z_{14}^*$ is a solution of the Schrödinger equation; $z_6^* + \mathrm{i}z_5^*$ and $z_{11}^* + \mathrm{i}z_9^*$ are solutions of mixed equations, diffusive in one variable and Schrödinger-like in the other variable; and $z_{15}^* + \mathrm{i}z_{16}^*$ is a solution of a mixed equation that is Schrödinger-like in one variable and conjugate Schrödinger-like in the other variable; the remaining $z_i^* = 0$.

3. Summary

The above calculations described a real physical probabilistic model. On the lattice the space of paths and the metric are both well known. The solutions of the discrete analogue of Schrödinger's equation are obtained by projection from the full solutions of the system. There is no formal analytic continuation involved in extracting these solutions; they are observable features of the ensembles of particles involved.

The full system is a microscopic model of quantum mechanics only in a formal sense, since the objects being described by solutions of Schrödinger's equation are ensembles of classical particles, and not the single 'particles' of nature. It is interesting to note, however, that if the Bohr–Einstein debate was about the above system, both arguments would contain a large element of truth. For example, in the $(1 + 1)$-dimensional system, Bohr would be correct in asserting that the ψ form a complete description. In this model they do form a complete description of a *subspace* of the full system. Furthermore, construction of the full system from the subspace is impossible because the information lost in the projection (z_1 and z_2) cannot be extracted from ψ.

On the other hand, Einstein's intuition that Schrödinger's equation is an incomplete description of the entire *physical* system would, in this case, be correct and to the point. From the above derivation we know that the information which is lost in the projection which reveals Schrödinger's equation, is crucial for the understanding of the context of the equation. For example, without all four components z_i in equation (12), the resulting Schrödinger equation (29) could be describing simple aspects of random walks, or it might just be a formal analytic continuation of a classical diffusive system, or it could be describing the particles of nature. Only the information, lost on projection, could distinguish these alternatives.

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